Name:……………………………………………………………………………………….

Student Number: ………………………………………………………………………..

Test 5 on WPPH16001.2017-2018
"Electricity and Magnetism"

Content: 10 pages (including this cover page)

Friday May 25 2018; A. Jacobshal 01, 9:00-11:00

- Write your full name and student number
- Write your answers in the designated area
- Reversed sides of each page are left blank intentionally and could be used for draft answers
- Read the questions carefully
- Please, do not use a pencil
- Books, notes, phones, and tablets are not allowed. Calculators and dictionaries are allowed.

Exam drafted by (name first examiner) Maxim S. Pchenitchnikov
Exam reviewed by (name second examiner) Steven Hoekstra

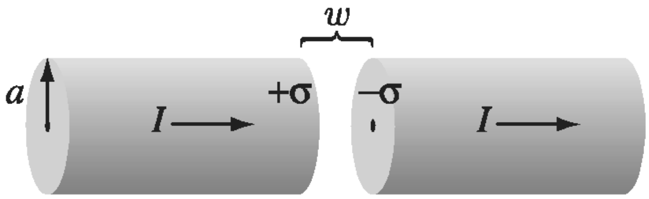
The weighting of the questions and the grading scheme (total number of points, and mention the number of points at each (sub)question).
Grade = 1 + 9 x (score/max score).

For administrative purposes; do NOT fill the table

	Maximum points	Points scored
Question 1	13	
Question 2	14	
Question 3	6	
Question 4	12	
Total	**45**	

Final mark: _________

Problem 1. (13 points)
A fat wire, radius a, carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel-plate capacitor, as shown in Figure.



As you remember from the previous test, the electric **E** and magnetic **B** fields inside the gap are given as

$$\mathbf{E}(t) = \boxed{\frac{It}{\pi\epsilon_0 a^2}\,\hat{\mathbf{z}}.} \qquad \mathbf{B}(s,t) = \boxed{\frac{\mu_0 I s}{2\pi a^2}\,\hat{\phi}.}$$

where z-axis is the horizontal symmetry axis that runs from the left to the right, s is the distance from the axis in the radial direction, t is time, and we assume that the charge is zero at t = 0.

1. Find the energy density u_{em} in the gap. (2 points)
2. Find the Poynting vector **S** in the gap. Note especially the direction of **S**. (2 points)
3. Show that the law of the local conservation of electromagnetic energy is satisfied. (3 points)
4. Determine the total energy in the gap, as a function of time. [If you're worried about the fringing fields, do it for a volume of radius $b < a$ well inside the gap.] (3 points)
5. Calculate the total power flowing into the gap, by integrating the Poynting vector over the appropriate surface. (2 points)
6. Show that the Poynting theorem is satisfied

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} u\, d\tau - \int_{\mathcal{S}} \mathbf{S}\cdot d\mathbf{a}$$

(Tip: W = 0 because there is no charge in the gap). (1 point)

Answer to Question 1 (Problem 8.2 from tutorials) 13 points

1. (2 points)

The energy density

$$u_{\rm em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \frac{1}{2}\left[\epsilon_0\left(\frac{It}{\pi\epsilon_0 a^2}\right)^2 + \frac{1}{\mu_0}\left(\frac{\mu_0 I s}{2\pi a^2}\right)^2\right] = \boxed{\frac{\mu_0 I^2}{2\pi^2 a^4}\left[(ct)^2 + (s/2)^2\right].}$$

2. (2 points)

The Poynting vector **S**

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0}\left(\frac{It}{\pi\epsilon_0 a^2}\right)\left(\frac{\mu_0 I s}{2\pi a^2}\right)(-\hat{\mathbf{s}}) = \boxed{-\frac{I^2 t}{2\pi^2\epsilon_0 a^4} s\,\hat{\mathbf{s}}.}$$

3. (3 points)

Local conservation of electromagnetic energy (from the formula sheet) $\frac{du}{dt} = -\boldsymbol{\nabla}\cdot\mathbf{S}$

$$\frac{\partial u_{\rm em}}{\partial t} = \frac{\mu_0 I^2}{2\pi^2 a^4}2c^2 t = \frac{I^2 t}{\pi^2\epsilon_0 a^4}; \quad -\boldsymbol{\nabla}\cdot\mathbf{S} = \frac{I^2 t}{2\pi^2\epsilon_0 a^4}\boldsymbol{\nabla}\cdot(s\,\hat{\mathbf{s}}) = \frac{I^2 t}{\pi^2\epsilon_0 a^4} = \frac{\partial u_{\rm em}}{\partial t}.$$

4. (3 points)

The total energy in the gap

$$U_{\rm em} = \int u_{\rm em} w 2\pi s\,ds = 2\pi w\frac{\mu_0 I^2}{2\pi^2 a^4}\int_0^b [(ct)^2 + (s/2)^2] s\,ds = \frac{\mu_0 w I^2}{\pi a^4}\left[(ct)^2\frac{s^2}{2} + \frac{1}{4}\frac{s^4}{4}\right]\Bigg|_0^b$$

$$= \boxed{\frac{\mu_0 w I^2 b^2}{2\pi a^4}\left[(ct)^2 + \frac{b^2}{8}\right].}$$

5. (2 points)

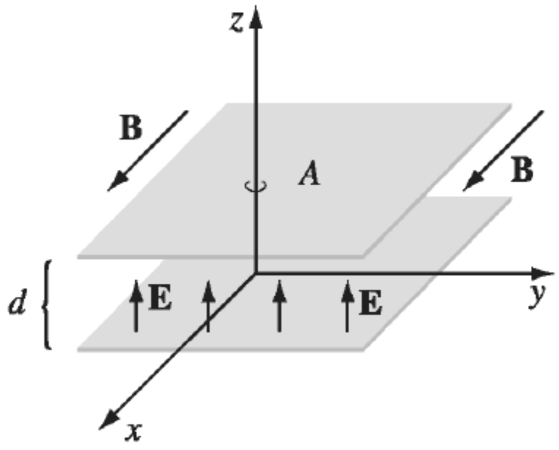
The total power flowing into the gap over a surface at radius b:

$$P_{\rm in} = -\int \mathbf{S}\cdot d\mathbf{a} = \frac{I^2 t}{2\pi^2\epsilon_0 a^4}[b\,\hat{\mathbf{s}}\cdot(2\pi b w\,\hat{\mathbf{s}})] = \boxed{\frac{I^2 w t b^2}{\pi\epsilon_0 a^4}.}$$

6. (1 point)

$$\frac{dU_{\rm em}}{dt} = \frac{\mu_0 w I^2 b^2}{2\pi a^4}2c^2 t = \frac{I^2 w t b^2}{\pi\epsilon_0 a^4} = P_{\rm in}$$

Question 2 (14 points)



A charged parallel-plate capacitor (with uniform electric field $\mathbf{E} = E\,\hat{\mathbf{z}}$) is placed in a uniform magnetic field $\mathbf{B} = B\,\hat{\mathbf{x}}$, as shown in Figure. A stands for the area of the plates; as usual, $d \ll \sqrt{A}$.

1. Determine all nine elements of the stress tensor $\overleftrightarrow{\mathbf{T}}$ in the region between the plates if $\mathbf{B} = \mathbf{0}$. Display your answer as a 3x3 matrix. (5 points)

2. From now on, $\mathbf{B} \neq \mathbf{0}$. Find the electromagnetic momentum in the space between the plates. (2 points)

3. Now a resistive wire is connected between the plates, along the z axis, so that the capacitor slowly discharges. The current through the wire will experience a magnetic force; what is the total impulse delivered to the system, during the discharge? Note that the charge is not given. (6 points)

4. Does the answer to #3 make any sense? (1 point)

Answer to Question 2 (Problem 8.7a from the tutorials and 8.6) 14 points

1. 5 points

$T_{xx} \equiv \epsilon_0 \left(-\frac{1}{2}E^2\right)$ (1 point)

$T_{yy} \equiv \epsilon_0 \left(-\frac{1}{2}E^2\right)$ (1 point)

$T_{zz} \equiv \epsilon_0 \left(E^2 - \frac{1}{2}E^2\right) = \frac{\epsilon_0}{2}E^2$ (1 point)

All other components equal zero because there are no cross-fields, i.e. $T_{xy} = \epsilon_0\left(E_x E_y\right) = 0$.

(1 point)

$\overleftrightarrow{\mathbf{T}} = \frac{\epsilon_0}{2}E^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (1 point)

2. 2 points

The momentum density: $\mathbf{g}_{\text{em}} = \epsilon_0(\mathbf{E} \times \mathbf{B}) = \epsilon_0 EB\,\hat{\mathbf{y}}$

The momentum: $\mathbf{P}_{\text{em}} = \epsilon_0 EBAd\,\hat{\mathbf{y}}$

3. 6 points

The total impulse delivered

$$\mathbf{P} = \int_0^\infty \mathbf{F}\,dt = \int_0^\infty I(t)\,(\mathbf{l} \times \mathbf{B})\,dt = \int_0^\infty I(t)Bd\,(\hat{\mathbf{z}} \times \hat{\mathbf{x}})\,dt = Bd\,\hat{\mathbf{y}} \int_0^\infty I(t)\,dt$$

$$= Bd\,\hat{\mathbf{y}} \int_0^\infty \left(-\frac{dQ(t)}{dt}\right) dt = -Bd\,\hat{\mathbf{y}}\,[Q(\infty) - Q(0)] = BdQ(0)\,\hat{\mathbf{y}}$$

But the original field was $E(t=0) = \frac{\sigma(t=0)}{\epsilon_0} = \frac{Q(0)}{\epsilon_0 A}; Q(0) = \epsilon_0 EA$

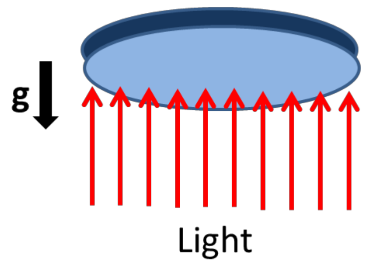
$\mathbf{P} = \epsilon_0 EBAd\,\hat{\mathbf{y}}$

4. 1 point

Yes, this is in line with the momentum conservation law.

Question 3. (6 points)

The intensity of sunlight hitting the earth is about 1300 W/m^2.
1. If sunlight strikes a perfect absorber, what pressure does it exert? Provide the value! (2 points)
2. How about a perfect reflector? Provide the value, too. (1 point)
3. What fraction of atmospheric pressure (10^5 N/m^2) does this amount to? (1 point)
4. What intensity is needed to levitate (i.e. to maintain at a steady position in the gravitational earth field, see Figure) a perfectly reflecting 1 gram disk of 1 cm radius? (2 points)



Answers to Question 3 (Problem 9.10 modified; also considered at lectures)

1. (2 points)

$$P = \frac{I}{c} = \frac{1.3 \times 10^3}{3.0 \times 10^8} = \boxed{4.3 \times 10^{-6}\, \mathrm{N/m^2}.}$$

2. (1 point)

For a perfect reflector the pressure is twice as great:

$$\boxed{8.6 \times 10^{-6}\, \mathrm{N/m^2}.}$$

3. (1 point)

Atmospheric pressure is 10^5 N/m^2

$$(8.6 \times 10^{-6})/(1.03 \times 10^5) = \boxed{8.3 \times 10^{-11} \text{ atmospheres.}}$$

4. (2 points)

$$mg = P \cdot \pi r^2 = 2\frac{I}{c}\pi r^2; I = \frac{mgc}{2\pi r^2}$$

$$I = \frac{10^{-3} \cdot 10 \cdot 3 \cdot 10^8}{2 \cdot 3 \cdot 10^{-4}} = 5 \cdot 10^9 \ \frac{W}{m^2}$$

<u>Question 4</u> (12 points)

An electromagnetic monochromatic plane wave with frequency ω is traveling in the direction from the origin to the point (1, 1, 1), with polarization parallel to the *xz* plane.

1. Give the explicit Cartesian components of $\mathbf{k}$ and $\hat{\mathbf{n}}$. (2 points)
2. Write down the (real) electric field for the wave of amplitude E_0 (consider the phase constant zero): $\tilde{\mathbf{E}}(\mathbf{r},t) = E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{n}}$ (4 points)

3. Write down the (real) magnetic field for the same wave. Don't forget to express B_0 via E_0.
$\tilde{\mathbf{B}}(\mathbf{r},t) = B_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\hat{\mathbf{k}} \times \hat{\mathbf{n}})$ (4 points)

4. Sketch the electromagnetic wave. (2 points)

Solution Question 4 (Griffiths 9.9 from tutorials) (12 points)

1. (2 points)

$$\mathbf{k} = \frac{\omega}{c}\left(\frac{\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}}{\sqrt{3}}\right)$$

Since $\hat{\mathbf{n}}$ is parallel to the *xz* plane, it must have the form $\hat{\mathbf{n}} = \alpha\hat{\mathbf{x}} + \beta\mathbf{z}$; since $\hat{\mathbf{n}} \cdot \mathbf{k} = 0$; $\beta = -\alpha$; and since $\hat{\mathbf{n}}$ is a unit vector, $\alpha = 1/\sqrt{2}$.

$$\hat{\mathbf{n}} = \frac{\hat{\mathbf{x}} - \hat{\mathbf{z}}}{\sqrt{2}}$$

2. (4 points)

$$\tilde{\mathbf{E}}(\mathbf{r}, t) = E_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\hat{\mathbf{n}}$$

$$\mathbf{k}\cdot\mathbf{r} = \frac{\omega}{\sqrt{3}c}(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}) \cdot (x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = \frac{\omega}{\sqrt{3}c}(x + y + z)$$

$$\mathbf{E}(x, y, z, t) = E_0 \cos\left[\frac{\omega}{\sqrt{3}c}(x + y + z) - \omega t\right]\left(\frac{\hat{\mathbf{x}} - \hat{\mathbf{z}}}{\sqrt{2}}\right)$$

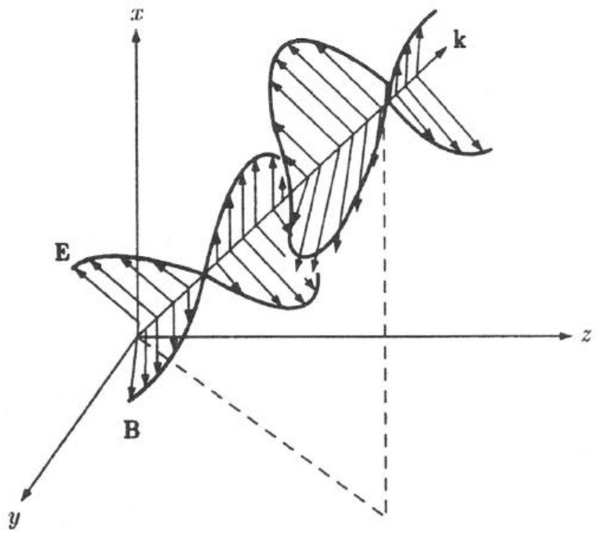
3. (4 points)

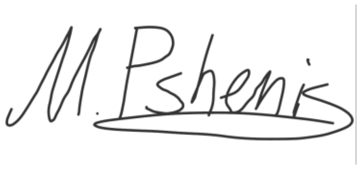
$$\tilde{\mathbf{B}}(\mathbf{r}, t) = \frac{1}{c}E_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}(\hat{\mathbf{k}} \times \hat{\mathbf{n}}) \text{ because } B_0 = \frac{1}{c}E_0$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{n}} = \frac{1}{\sqrt{6}}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \frac{1}{\sqrt{6}}(-\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}} - \hat{\mathbf{z}})$$

$$\mathbf{B}(x, y, z, t) = \frac{E_0}{c}\cos\left[\frac{\omega}{\sqrt{3}c}(x + y + z) - \omega t\right]\left(\frac{-\hat{\mathbf{x}} + 2\hat{\mathbf{y}} - \hat{\mathbf{z}}}{\sqrt{6}}\right)$$

4. (2 points)





Maxim Pchenitchnikov
May 19 2018

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